The problem of scale separation in granular flows driven by gravity

Giulia Rossi[a], Aronne Armanini[a]

[a]*Department of Civil, Environmental and Mechanical Engineering, University of Trento, Via Mesiano, 77 - 38123 Trento, Italy.*

Abstract

Starting from the observation that granular flows driven by gravity can be considered a macroscopic representation of molecular gases, kinetic theories have been applied to granular flows since 1978 (Ogawa).
It must be highlighted that in molecular gases there is a strong scale separation: the microscale, represented by the mean free paths, is very much smaller than the macroscale, that is the scale according to which gradients change (Goldhirsch 2003, 2007). In granular flows, we still have a scale separation, but not so strong: the dimension of a single particle becomes comparable to that of the control volume. This leads probably to a correlation between the single particle velocity and the concentration, hence the definition of the averaged values becomes crucial in this respect. We may distinguish between two types of average: the ensamble average, used in the standard kinetic theories, in which we consider constant the number of particles per unit n (that represents the concentration) and the average defined by Drew, where we should consider a variable n. This leads to this inequality: $< nc > \neq n < c >$, and the derivatives of such average may have additional diffusive terms, currently not present in the continuity equation.

Keywords: granular flows, kinetic theories, Boltzmann equation, scale separation

1. Intoduction

Since many years granular flows have been compared to molecular gases and studied through the kinetic theories. The collisions among particles have been assimilated to that among molecules in gases and the pressure is considered as a consequence of them. The single particle characteristics are described by the *distribution function*, which recalls the Gaussian distribution and represents what percentage of particles are in a certain zone of the control volume within a certain range of velocity; only *binary* and *instantaneous* collision are expected to occur and their distribution function are correlated through a function $g(r)$, called *radial distribution function*.
Kinetic theories are based on the *Boltzmann equation*, which expresses the conservation law for the properties of the flow; it is based on the concept of *ensamble average*, that is the average done over all the particles of the ensamble. Following its mathematical definition, we have noticed that not all assumptions behind it are valid for the granular flows case: the lack of strong

*Corresponding author
Email addresses: giulia.rossi@unitn.it (Giulia Rossi), aronne.armanini@unitn.it (Aronne Armanini)

scale separation in granular flows, leads to some crucial differences between molecular gases and granular flows, for which we suppose a new type of average is needed and, consequently, new terms in the continuity equation will appear.

2. Problem of scale separation

The problem of the lack of scale separation in granular flows have been already pointed out in the past (see [3], [4], [6]) and we are trying to study and understand in deep its meaning.
Analyzing the definition of the number density (which represents the volume fraction concentration), we can say that $n = f(\mathbf{r}, t)$, i.e. it doesn't depend on the velocity, because by definition it is the integral over all the velocities of the distribution function:

$$n(\mathbf{r}, t) = \int f^{(1)}(\mathbf{r}, \mathbf{c}, t) d\mathbf{c} \tag{1}$$

Since the *ensamble average* of a general property of the granular gas is defined as follows:

$$< \psi >= \frac{1}{n} \int \psi f^{(1)}(\mathbf{r}, \mathbf{c}, t) d\mathbf{c} \tag{2}$$

then we can conclude that if we have an ensamble average of a property multiplied by n, this one could be considered as a constant in the integral and so we obtain that:

$$< n\psi >= \frac{1}{n} \int n\psi f^{(1)}(\mathbf{r}, \mathbf{c}, t) d\mathbf{c} = \frac{n}{n} \int \psi f^{(1)}(\mathbf{r}, \mathbf{c}, t) d\mathbf{c} = \int \psi f^{(1)}(\mathbf{r}, \mathbf{c}, t) d\mathbf{c} = n < \psi > \tag{3}$$

Indeed, since n doesn't depend on $\mathbf{c}$, it can be brought out of the integral.
Starting from these equations, we have to consider that kinetic theories were born for molecular gases, in which it is assumed that the control volume is taken large enough to contain a very high number of particles ($n \to \infty$), but small enough so that compared to the macroscopic dimensions it may be regarded as a point (see [5]).
Clearly this choice is no more possible for granular flows: the control volume dimension becomes comparable to that of particles and this leads to some important consequences. In particular, equations (1), (2) and (3) are correct only in case of a strong scale separation, in which the concentration, represented by n, could be considered constant in more realizations; on the contrary in granular flows, this procedure is applicable only to a **single realization**, as pointed out by Goldhirsh [3]. The central concept is that granular flow are not *ergodic systems*, as the gases are, for the following reasons:

1. In gases the number density n could be considered constant, since the number of particles tends to infinity and the concentration is not affected significantly by one particle more or less. On the contrary in granular flows, where we have a control volume whose dimensions are comparable to that of particles, one particle more or less may change significantly the value of n.
2. The presence of gradients at the same scale of particles collisions, may cause a non-uniform flow along the vertical axis (contrary to the longitudinal axis) so that one realisation could be not representative of the whole process, regarding the vertical direction.

Following these arguments, we may say that the kinetic theories average process, based on the ensamble average, could be applied in first approximation to a single realization in granular flows field. For this purpose we assume that, from a discrete point of view, the ensamble average done on a single realization is computed as:

$$< \psi >\simeq \psi_k = \frac{\sum_1^{n_k} \psi_s}{n_k} \tag{4}$$

where ψ_k is the average of a single realisation, ψ_s is the property of the single particles, n_k is the number of particles in the k-th realization and k is the realization indices.
On the other hand, for more that one realization we need a new type of averaging process. We decided to define the average of a property in more than one realization as the simple aritmetic average over all the single realizations ensamble average:

$$\overline{\psi} = \frac{\sum_1^{R} \psi_k}{R} \tag{5}$$

where $\overline{\psi}$ is the average of a general property, ψ_k is the ensamble average of the single realization computed in equation (4) and R is the number of realization on which we average.
Finally, since n is no more constant in all realizations, we may define an average value of it, as follows:

$$\overline{n} = \frac{\sum_1^{R} n_k}{R} \tag{6}$$

where n_k is the number density of the k-th realization.
We want to underline that in making these averages, we assume that the fluctuations with respect to the mean values of the single realization are governed by collisional mechanisms, related to the kinetic theories; on the other hand, in the case of several realizations, the fluctuations are related to macroscopic scales (like the flow depth or the mean velocity of the flow).

3. New formulation of the continuity equation

The usual continuity equation considered is the following:

$$\frac{\partial < nm >}{\partial t} + \frac{\partial < nmc_y >}{\partial y} + \frac{\partial < nmc_y >}{\partial x} = 0 \tag{7}$$

where m represents the mass, c_y the y-component velocity and nm is the density. Considering a statistically stationary and uniform flow in the longitudinal direction x, equation (7) reduces to:

$$\frac{\partial < nmc_y >}{\partial y} = 0 \tag{8}$$

This implies that $< c_y >$ is equal to zero, that is no vertical component of the mean motion is present.
In the case of granular case, since the system is not ergodic, the average of a single realization is not the same of that of more realizations: $< n >$ is not the single realization n, but $\overline{n}$ (an average on more than one realization defined in equation (6)), and $< nc_y >$ is a more complex term, since it is no more the simple product of the mean of n and the mean of c_y. Indeed, the 'fluctuations'

contribute must be taken into account in this type of flow; expressing the continuity equation for R realizations as the sum of the single-realization continuity equation, we obtain:

$$\frac{\sum_1^R \frac{\partial}{\partial y}(c_k n_k)}{R} = 0 \tag{9}$$

We can rewrite the previous equation as follows:

$$\frac{\partial}{\partial y}(\overline{c_k n_k}) = 0 \tag{10}$$

where the overlined term indicates the average over all the R realizations. Afterwards, we express u_k and n_k as the sum of a mean value and of a fluctuation term:

$$c_k = \overline{c} + c'_k, \qquad n_k = \overline{n} + n'_k \tag{11}$$

where $\overline{c}$ is the average type defined by equation (5) and $\overline{n}$ by equation (6).
Inserting these terms in the continuity equation (10) and averaging, we obtain the following equation:

$$\frac{\partial}{\partial y}(\overline{n} \quad \overline{c}) + \frac{\partial}{\partial y}\overline{n'_k c'_k} = 0 \tag{12}$$

since by definition $\overline{u'_k} = \overline{n'_k} = 0$. This equation implies that $\overline{c_y} \neq 0$; then it follows that:

$$\overline{n} \quad \overline{c_y} = -\overline{n' c'_y} \tag{13}$$

4. Experimental analysis

In order to verify our previous considerations, we have carried out some experiments on dry granular flow in the Hydraulic Laboratory of the University of Trento. We used a glass-walled open channel of about 3 m lenght and 5 cm width; through a weir in the downstream end of the channel, a deposition layer is created (static bed). The dry granular material flows over it and we have mobile bed condition; in this condition, with a constant discharge in time, we have a stationary regime [1], and the free surface slope and mobile bed slope coincide.
Videos of the flow have been acquired through two cameras with a video rate of 2000 fps, one at the sidewall and one above the flow free surface. The material used is made up of zeolites of almost spherical shape, whose diameter has been chosen to be between 0.5 and 0.6 mm. The velocity of the particle and their concentration were derived through the Voronoi method [7].
Within this procedure, we have indentified, in our statistically stationary process, a single realization as a record long enough to catch a whole collisional event, but short enough with respect to the fluctuations linked to the spatial gradient.
The results of our experiments were quite interesting, since, by considering a steady uniform flow (the derivatives with respect to t and x are zero), the value of $\overline{c}$ is not zero. This seems to be consistent with our theoretical considerations; furthermore we are analysing the transversal velocity component at the top of the flow in order to evaluate the presence of vortices and compare it with the vertical component we have noticed.

5. Conclusions

We analized the hypothesis behind the kinetic theories and the Boltzmann equation, noticing that some assumptions do not hold for granular flows. In particular we focused our attention on the problem of the lack of strong scale separation due to the fact that granular systems are not ergodic, contrary to the molecular case. Because of that, the *ensamble average* may be applied to a single realization, while, for more than one realization, it would be more correct to use a different avaraging process (equation (5)). Considering a new type of average, and averaging the variables considering the mean contribution and the fluctuations contributions, a new form of the continuity equation has been obtained (equation (12)).

Our future challenge, is to define a model to describe the larger fluctuations (larger with respect to the collision scale): we suppose to elaborate it with a diffusive model, whose scale will be probably proportional to the flow depth h and to the normalized velocity u_* and perhaps other terms.

Furthermore, we will carry out much more experiments in various conditions (varying the flow rate, the density, the flow height) in order to evaluate more precisely the vertical component of the motion and to identify the contribution of possible secondary circulations.

Finally, we will extend this arguments to the moment equation.

6. Bibliography

[1] Armanini A., Capart H., Fraccarollo L., Larcher M. (2003) *Debris flow. Encyclopedia of hydrological science*,Anderson MG, Ed: John Wiley & Sons.

[2] Drew D.A.(1983) *Mathematical modelling of two-phase flow*, Annu. Rev. Fluid Mech. vol. 15, pp.261-291.

[3] Goldhirsch I. (2003) *Rapid Granular Flow*, Annu. Rev. Fluid. Mech. vol. 35, pp.267-293.

[4] Goldhirsch I. (2008) *Introduction to granular temperature*, Powder Technology. vol. 182, pp.130-136.

[5] Huang K. (1928) *Statistical Mechanics*, Massachusetts Institute of Technology. Ed: John Wiley & Sons.

[6] Jaeger H.M., Nagel S.R., Behringer R.P. (1996) *Granular solids, liquids and gases*, Reviews of Modern Physics, vol. 68, pp. 1259-1273.

[7] Spinewine B., Capart H., Larcher M., Zech Y. (2003) *Three-dimensional Voronoi imaging method for the measurement of near-wall particulate flow*,Exp Fluids. vol 34, pp. 227-241.